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MASS TRANSFER THROUGH LAMINAR BOUNDARY LAYERS—3a. SIMILAR SOLUTIONS OF THE b-EQUATION WHEN $B = 0$ AND $\sigma \geqslant 0{\cdot}5$

H. L. EVANS

Commonwealth Scientific and Industrial Research Organization,
Division of Food Preservation, Homebush, N.S.W., Australia*

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Abstract—Using a form of expansion first suggested by Merk, series are given for evaluating accurately the "wall gradient" (b_0'/B) for the case when no mass flows through the phase boundary. These cover wide ranges of main-stream pressure gradient and give good accuracy for any value greater than 0·5 of the Prandtl/Schmidt number σ. Values obtained compare well both with the few exact solutions given in the literature and with exact numerical integration. Curves are drawn of a number of other functions obtained from this "wall gradient".

Résumé—Des séries, obtenues en utilisant une forme de développement préconisée primitivement par Merk, sont données pour le calcul précis du "gradient à la paroi" b_0/B dans le cas où aucun transport de masse ne s'effectue à la limite de la phase. Elles couvrent un grand domaine de gradient de pression de l'écoulement principal et assurent une bonne précision pour toute valeur σ du rapport du nombre de Prandtl au nombre de Schmidt supérieure à 0,5. Les valeurs obtenues se comparent bien aux quelques solutions exactes données dans la littérature et à celles de l'intégration numérique exacte. Des sourbes d'un certain nombre d'autres fonctions obtenues à partir de ce "gradient à la paroi" sont tracées.

Zusammenfassung—Mit Hilfe einer zuerst von Merk vorgeschlagenen Reihenentwicklung kann der „Wandgradient" (b_0'/B) genau berechnet werden, wenn kein Mengenstrom durch die Phasentrennschicht tritt. Diese Reihen umfassen ein grosses Gebiet des Druckgradienten der Hauptströmung und sind für beliebige Werte der Prandtl/Schmidtzahl σ die grösser als 0,5 sind, sehr genau. Die errechneten Werte stimmen gut überein mit den wenigen genauen Lösungen der Literatur und der exakten numerischen Integration. Kurven für eine Reihe anderer, mit Hilfe dieses „Wandgradienten" erhaltener Funktionen sind angegeben.

Аннотация—Используя вид разложения, впервые предложенный Мерком, приводятся ряды для точной оценки «пристенного градиента» (b_0/B) для случая, когда отсутствует поток массы через границу фазы. Эти разложения охватывают широкие области градиента давления основного потока и дают высокую точность во всех случаях, когда отношение числа Прандтля к числу Шмидта σ больше 0,5. Полученные величины хорошо сопоставимы как с некоторыми имеющимися в литературе точными решениями, так и с результатами точного численного интегрирования. Приводятся кривые некоторых других функций, полученных на основе «пристенного градиента».

NOTATION

- a_q Coefficients in equation (13); see also Table 2 (—);
- A_n Coefficients occurring in equation (9) and defined in equation (11) (—);
- b Dimensionless conserved fluid property (see Paper 3 for discussion of its form and meaning) (—);
- b_0' Gradient of b adjacent to the interface (—);
- B Driving force for mass transfer (discussed in Paper 3) (—);
- C_m Coefficients occurring in equation (18) and defined by equation (20) (—);
- E Integral in equation (9); evaluated numerically using Table 2 (—);
- E_{sep} Integral of equation (18); evaluated numerically from equation (21) (—);

* The Division is now located at P.O. Box 43, Ryde, N.S.W., Australia.

f Dimensionless stream function (defined and discussed in Paper 1) (—);

$f_0, f'_0, f''_0, f'''_0, f_0^{(n)}$ The value of f and its derivatives at the interface (—);

$F_2 \left(= \frac{u_G}{\nu} \frac{d\delta_2^2}{dx} \right)$ parameter giving rate of growth of the momentum thickness δ_2; discussed in Paper 1 (—);

g Mass transfer conductance (see Paper 3) (lb/ft^2h);

g^* Value of g for $B = 0$. (See equation 27) (lb/ft^2h);

m Number occurring in equation (20) specifying terms in equation (18). Also used to specify terms in equation (9) like that of equation (12) (—);

n Number occurring in equation (11) specifying terms in equation (9) (—);

Nu Local Nusselt number in terms of the length x (—);

q Number specifying terms in equation (13) (—);

r Number occurring in equation (15) (—);

Re Local Reynolds number ($= u_G x/\nu$) (—);

u Velocity component parallel to interface (ft/h);

u_G Value of u in the main-stream immediately outside the boundary layer (ft/h);

v Velocity component normal to interface (ft/h);

W Curvature parameter in terms of Δ_2 (equation 30) (—);

x Distance parallel to interface measured from start of boundary layer (ft);

X Curvature parameter in terms of Δ_4 (equation 28) (—);

X_{sep} Curvature parameter in terms of Δ_4 applicable to separation point (equation 39) (—);

y Distance perpendicular to interface (ft);

Y Parameter which is a measure of the rate of growth of Δ_4 (equation 29) (—);

Y_{sep} Parameter corresponding to Y applicable near separation point (equation 38) (—);

Z Parameter which is a measure of the rate of growth of Δ_2 (equation 31) (—).

Greek symbols

β Parameter occurring in velocity equation (equation 2); discussed in Paper 1 (—);

γ Fluid property called the "exchange coefficient" (diffusion coefficient times density, or thermal conductivity divided by specific heat at constant pressure) (lb/ft h);

δ_2 Momentum boundary layer thickness $= \int_0^\infty (u/u_G)(1 - u/u_G)\, dy$ (ft);

δ_4 Shear boundary layer thickness $= u_G/(\partial u/\partial y)_0$ (ft);

Δ_2 Convection boundary layer thickness $= \int_0^\infty (u/u_G)(1 - b/B)\, dy$ (ft);

Δ_4 Conduction boundary layer thickness $= B/(\partial b/\partial y)_0$ (ft);

η Dimensionless distance variable for "similar" solutions (—);

μ Dynamic viscosity of fluid (lb/ft h);

ν Kinematic viscosity of fluid ($= \mu/\rho$) (ft^2/h);

ρ Density of fluid (lb/ft^3);

σ Prandtl or Schmidt number ($= \mu/\gamma$) (—);

φ Integration variable occurring in equations (9) and (18) (—).

Subscripts

G Denotes fluid state in main-stream just outside the boundary layer;

sep Denotes "separation" conditions of section 3.2;

0 Denotes fluid state adjacent to the interface;

q Denotes terms in equation (13);

m Denotes terms in equation (18);

n Denotes terms in equation (9).

A prime $'$ denotes differentiation with respect to η.

1. INTRODUCTION

1.1 *General remarks*

The first two papers in the present series, Spalding [1], Spalding and Evans [2], were concerned exclusively with the velocity equation of laminar boundary layer theory when mass flows through the boundary layer, and Paper 3, Spalding and Evans [3], dealt with the b-equation. In this latter paper tables were given from which new approximate solutions to the b-equation could be evaluated. These gave reasonable agreement with known exact solutions over wide ranges of the Prandtl/Schmidt number, main-stream pressure gradient and the mass transfer driving force B.

The present paper is concerned with the case when B, the driving force for mass transfer, is zero. The work has developed out of the methods employed in obtaining the earlier results so frequent reference will be made to the first three papers in the series.

Since the main aim of the present series is the prediction of mass transfer rates, the case when mass transfer is in fact zero may be thought to be irrelevant. But this is not the case, for, as was pointed out in Paper 3, we are interested in families of solutions to the boundary layer equations for both positive and negative values of B; the solutions for $B = 0$ must therefore form important members of these families. It is also found that many practical problems involving mass transfer correspond to very small values of B; this is often the case, for example, when water vapour either evaporates into or condenses out of the atmosphere at ordinary temperatures and pressures. The case for $B = 0$ can then serve as a useful first approximation, the effects of mass transfer being regarded as a small correction. In fact, Paper 3 contained an approximate formula for estimating the mass transfer conductance g from the value of g^*, the conductance when $B = 0$. Using tables to be given later it is possible to calculate g^* to high accuracy.

The present results are expected to be most useful, however, in problems when mass transfer is entirely absent; the most obvious case is that of heat transfer to or from an impervious wall. Although, in the literature, a number of papers consider this type of problem, comparatively few exact solutions to the equations could be found; most of these have already been quoted in Paper 3.

The quantity being evaluated in the present paper is (b'_0/B). Reference to formulae in Paper 3 indicates that when $B = 0$, so is b'_0 $(= -\sigma f_0)$. The ratio (b'_0/B) is not zero, however, for, in terms of the familiar problem of pure heat transfer it is the gradient at the wall of the dimensionless temperature with respect to the "similar" distance variable η. In the customary nomenclature of heat transfer, therefore, we have:

$$(b'_0/B) = Nu\,(2 - \beta)^{1/2}/Re^{1/2}. \tag{1}$$

where:

Nu = local Nusselt number,

Re = local Reynolds number, and

β = parameter occurring in the velocity equation.

1.2 *Outline of present paper*

The main purpose of the present paper is to give series expansions from which, knowing the solutions to the "similar" velocity equation, values of the "wall gradient" (b'_0/B) may be obtained to high accuracy for a range of values of the parameter β, which determines the magnitude of the main-stream pressure gradient, and for any value greater than 0·5 of the fluid property group σ.

The series from which values of (b'_0/B) are obtained are in increasing inverse powers of σ; they are therefore very accurate for high values of σ. Some exact values for $\sigma = 0{\cdot}7$ were found in the literature. Sufficient terms were therefore taken in the series to give close agreement in the fourth significant figure with these exact solutions. The values of (b'_0/B) quoted are then accurate to better than 0·1 per cent even when σ is as low as 0·7.

Using the series, a table of (b'_0/B) as a function of the parameters β and σ has been drawn up in which the interval in both these parameters is small enough to allow of rapid interpolation for intermediate values. In many cases, however, it may be more satisfactory to use the original series for intermediate values of σ.

As has been pointed out in earlier papers, the main interest in obtaining "similar" solutions to the boundary layer equations is not so much in their application to strictly "similar" physical configurations (e.g. flow over wedges) as in the way they can be used to solve more general problems involving non-similar flows.

Some functions which are useful for such calculations are plotted in Figs. 1–4 but are only briefly discussed here as their significance and applications are to be considered more fully in other papers in the series.

2. STATEMENT OF THE MATHEMATICAL PROBLEM

2.1 *Introduction*

It was shown in Paper 3 that the b-equation governs the distribution in the boundary layer of any conserved fluid property. The present paper is mainly concerned with the "similar" solutions to this equation. It should be made clear, however, that the word "solution" is used in a restricted sense in that the important entity to be evaluated is the gradient of b at the phase boundary, although many other functions in the boundary layer may be obtained from this gradient. Interest is concentrated on this "wall gradient" because the present series of papers is mainly concerned with rates of transfer through the phase boundary, these rates being determined immediately the wall gradient is known. The distribution of b in the boundary layer is not evaluated, although obtaining the wall gradient is an important first step in getting this distribution.

Solutions to the b-equation depend, of course, on the distribution of the stream function. It will therefore be necessary to refer frequently to the velocity equation for "similar" flows. This will be quoted but not discussed; reference to discussions and tables given in Papers 1 and 2 will suffice for the purposes of the present paper.

2.2 *The velocity equation*

In Paper 1, Spalding [1], it was shown that, for "similar" solutions to the boundary layer equations the velocity distribution is governed by the ordinary differential equation:

$$f''' + ff'' + \beta(1 - f'^2) = 0. \tag{2}$$

When no mass flows through the interface, the boundary conditions associated with this equation are:

$$\left.\begin{array}{l} \eta = 0,\ f = 0,\ f' = 0 \\ \eta \to \infty,\ f' \to 1 \end{array}\right\}. \tag{3}$$

In equations (2) and (3) the primes denote differentiation with respect to the dimensionless space co-ordinate η, f is the dimensionless stream function and β a parameter whose value depends on the acceleration of the main-stream. These functions are more fully defined in the notation list.

2.3 *The b-equation*

In Paper 3, Spalding and Evans [3], it was shown that for a certain restricted group of "similar" solutions to the boundary layer equations, the distribution of a function b, which represents a conserved fluid property in suitable dimensionless form, is governed by the ordinary differential equation:

$$b'' + \sigma f b' = 0 \tag{4}$$

with the boundary conditions:

$$\left.\begin{array}{l} \eta = 0,\ b = 0 \\ \eta \to \infty,\ b \to B \end{array}\right\}. \tag{5}$$

In equation (4) the primes again denote differentiation with respect to η, f is the stream function occurring in equation (2) and σ is the fluid property group called the Prandtl or Schmidt number depending on the transferred fluid property which is under investigation. The quantity B occurring in equation (5) is the value of b in the main-stream and is assumed to be constant.

From equation (4) the gradient of b at the interface, namely b_0', can in principle be evaluated once solutions to the velocity equation are known. This is done using the relationship:

$$(B/b_0') = \int_0^\infty \exp -\{\sigma \int_0^\eta f\,d\eta\}\,d\eta. \tag{6}$$

It is convenient, for the present, to work in terms of the reciprocal of (b_0'/B).

The aim of the present paper is to evaluate this integral for a range of values of the parameter β occurring in equation (2) and for any value of σ greater than 0·5.

The tables given in Paper 3 included the case $B = 0$ and approximate values of this integral could be obtained from them. The accuracy of those tables increased with increasing σ. The present results are accurate for all high values of σ.

3. EVALUATING THE INTEGRAL AS A POWER SERIES IN σ

When evaluating the integral in equation (6) for various values of the parameter β, with the single exception of the separation point when $\beta = -0{\cdot}1988$, all values of β considered here could be treated by one general method. This will be given first and the separation point will then be treated by a modified form of this method.

3.1 *The general case*

It is clear that the integral in equation (6) for any value of σ depends only on the distribution with η of the stream function f. Unfortunately, however, only a few accurate values of the integral could be found in the literature.

On the other hand Merk [4] has shown that the integral can be expressed as a series in inverse powers of σ, which would be very accurate for large values of σ. Although the present work arose out of that paper, the method of deriving the series expansion differs in many respects from that given by Merk, and so will be given in outline below. It has also been possible to modify the present method to include cases when mass transfer occurs; this work is to be considered elsewhere.

Denoting quantities when evaluated at the interface by the suffix 0, the stream function f can be expanded in terms of derivatives there in the form:

$$f = \frac{f''_0\eta^2}{2!} + \frac{f'''_0\eta^3}{3!} + \frac{f^{\text{iv}}_0\eta^4}{4!} + \frac{f^{\text{v}}_0\eta^5}{5!} + \dots \quad (7)$$

Inserting this into the integral in equation (6) and changing the variable from η to φ, where:

$$\varphi = \frac{\sigma f''_0}{3!}\eta^3 \quad (8)$$

the integral to be evaluated becomes:

$$E = \int_0^\infty e^{-\varphi}\varphi^{-2/3} \exp -\{A_3\varphi^{4/3} + A_5\varphi^{6/3} + A_6\varphi^{7/3} + \dots\} \, \mathrm{d}\varphi. \quad (9)$$

The quantity (b'_0/B) is obtained from E using the relationship:

$$(b'_0/B) = \frac{3}{E}\left(\frac{\sigma f''_0}{3!}\right)^{1/3} \quad (10)$$

since f''_0 is known. The problem therefore reduces to the evaluation of E in equation (9).

The tables given in Paper 3 were approximate in that only the first two terms in the expansion given in equation (7) were used (remembering that we are considering here the case of no mass transfer so that f_0 is zero). In the present paper many more terms are included.

The coefficients A_n occurring in the expression for E are given by the general formula:

$$A_n = \frac{1}{\sigma^{(n-2)/3}}\frac{f_0^{(n)}}{(n+1)!}\left(\frac{3!}{f''_0}\right)^{(n+1)/3}. \quad (11)$$

Since the present discussion is restricted to the case when no mass flows through the interface the coefficients A_0, A_1 and A_4 are zero, and because of the transformation used in obtaining E, the coefficient A_2 is reduced to unity.

In order to calculate the coefficients A_n the values of $f_0^{(n)}$, the derivatives of f at the wall, were needed. These were obtained from equation (2) by successive differentiation. In this way each $f_0^{(n)}$ higher than f''_0 could be expressed in terms of f''_0 and the parameter β.

Values of the function f''_0 for the values of β considered in the present paper are given in Table 1. For values of β up to 2·0 these were taken from Falkner [5, 6]; the method used for evaluating f''_0 for higher values of β was given in Paper 2.

In the expression for E given in equation (9) the second exponential term in the integrand can be expanded as a series in powers of φ. Each term under the integral sign is then of the form $e^{-\varphi}\varphi^{m/3}$, where m has a different value for each term; the integral may then be readily evaluated term by term using the relationship:

$$\int_0^\infty e^{-\varphi}\varphi^{m/3}\,\mathrm{d}\varphi = \Gamma\left(\frac{m+3}{3}\right) \quad (12)$$

Table 1. Values of the stream function "wall gradient" f''_0 for the values of β considered in the present paper

β	−0·1988	−0·1	0·0	0·1	0·2	0·3	0·4	0·5
f''_0	0	0·3191	0·4696	0·5870	0·6869	0·7748	0·8542	0·9277
β	0·6	0·8	1·0	1·4	2·0	3·0	4·0	5·0
f''_0	0·9960	1·1200	1·2326	1·431	1·687	2·043	2·346	2·614

where Γ denotes the gamma function. The final step in obtaining the required series for E is to rearrange these terms into a series in inverse powers of σ having the form:

$$E = \Gamma(\tfrac{1}{3}) + \sum_{q=1}^{\infty} \frac{a_q}{\sigma^{q/3}} \tag{13}$$

in which the coefficients a_q are functions of the quantities A_n and gamma functions; these coefficients become more complicated as q increases.

The numerical values of only three gamma functions were needed, namely:

$$\Gamma(\tfrac{1}{3}) = 2{\cdot}67894,\ \Gamma(\tfrac{2}{3}) = 1{\cdot}35412,\quad \Gamma(1) = 1. \tag{14}$$

All other gamma functions which occurred could be expressed in terms of these using the relationship:

$$\Gamma(r+1) = r\Gamma(r) \tag{15}$$

which holds for any value of r for which the gamma function exists.

Numerical values of the first nine coefficients a_q are given in Table 2. This covers the range $-0{\cdot}1 \leqslant \beta \leqslant 5{\cdot}0$. In both directions outside this range the accuracy of the present method diminished, particularly when σ was less than unity. The region approaching conditions leading to "separation" of the velocity boundary layer, namely for $-0{\cdot}1988 \leqslant \beta \leqslant -0{\cdot}1$, was particularly difficult to treat. On the other hand the separation point itself, when $\beta = -0{\cdot}1988$, yielded a series which gave high accuracy.

3.2 *The separation point*

When $\beta = -0{\cdot}1988$ the velocity boundary layer is said to "separate" from the interface; for the present discussion this merely means that f''_0 becomes zero. The above analysis does not then apply because the transformation defined by equation (7) cannot be used. A modification of the procedure can, however, lead to an asymptotic series from which (b'_0/B) can be evaluated to high accuracy. Merk [4] also considered this case but as there appears to be a numerical error in the expansion he obtained, the derivation of the relevant series will be given here.

Table 2. Values of the first nine coefficients in the series expansion of the integral E.

$$\left(\frac{b'_0}{B}\right) = \frac{3}{E}\left(\frac{\sigma f''_0}{3!}\right)^{1/3}; \qquad E = 2{\cdot}67894 + \sum_{q=1}^{\infty} \frac{a_q}{\sigma^{q/3}}$$

β	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
−0·1	−0·18808	0·04341	0·05748	−0·00357	0·00037	−0·00702	−0·00185	−0·00098	0·000968
0·0	0·0	0·0	0·05953	0·0	0·0	−0·00661	0·0	0·0	0·000717
0·1	0·08344	0·00854	0·04884	0·00153	−0·00001	−0·00587	−0·00062	−0·00013	0·00089
0·2	0·13533	0·02247	0·04092	0·00144	−0·00045	−0·00516	−0·00081	−0·00021	0·00092
0·3	0·17289	0·03668	0·03465	0·00110	−0·00118	−0·00456	−0·00086	−0·00018	0·00091
0·4	0·20240	0·05027	0·02930	0·00080	−0·00206	−0·00403	−0·00085	−0·00002	0·00090
0·5	0·22663	0·06302	0·02442	0·00057	−0·00302	−0·00352	−0·00086	0·00019	0·00086
0·6	0·24738	0·07509	0·01985	0·00042	−0·00403	−0·00301	−0·00089	0·00045	0·00086
0·8	0·28207	0·09763	0·01136	0·00040	−0·00612	−0·00189	−0·00103	0·00109	0·00056
1·0	0·31031	0·11816	0·00316	0·00053	−0·00834	−0·00061	−0·00127	0·00190	0·00018
1·4	0 35604	0·15555	−0·01247	0·00140	−0·01296	0·00262	−0·00201	0·00390	−0·00111
2·0	0·40841	0·20468	−0·03568	0·00315	−0·02078	0·00913	−0·00390	0·00787	−0·00464
3·0	0·47470	0·27651	−0·07325	0·00774	−0·03527	0·02523	−0·00889	0·01731	−0·01553
4·0	0·52632	0·33992	−0·11085	0·01306	−0·05227	0·04740	−0·01673	0·03016	−0·03392
5·0	0·56951	0·39798	−0·14828	0·01907	−0·07133	0·07600	−0·02728	0·04692	−0·06080

When $\beta = -0{\cdot}1988$, $\left(\frac{b'_0}{B}\right) = \frac{4}{E_{sep}}\left(\frac{\sigma f'''_0}{4!}\right)^{1/4}$; $E_{sep} = 3{\cdot}62561 + \frac{0{\cdot}084049}{\sigma} - \frac{0{\cdot}005907}{\sigma^2} - \frac{0{\cdot}0001405}{\sigma^3} + 0(\sigma^{-4})$

Since f''_0 is zero the expansion for f takes the form:

$$f = f'''_0 \frac{\eta^3}{3!} + f^{\mathrm{VII}}_0 \frac{\eta^7}{7!} + f^{\mathrm{XI}}_0 \frac{\eta^{11}}{11!} + f^{\mathrm{XV}}_0 \frac{\eta^{15}}{15!} + \ldots \quad (16)$$

Using the transformation:

$$\varphi = \left(\frac{\sigma f'''_0}{4!}\right) \eta^4, \quad (17)$$

the integral to be evaluated, denoted by E_{sep} since it applies only to the separation point, is:

$$E_{sep} = \int_0^\infty e^{-\varphi}\varphi^{-3/4} \exp -\{C_7\varphi^2 + C_{11}\varphi^3 + C_{15}\varphi^4 + \ldots\} \, d\varphi \quad (18)$$

and (b'_0/B) is obtained from E_{sep} using the relationship:

$$(b'_0/B) = \frac{4}{E_{sep}} \left(\frac{\sigma f'''_0}{4!}\right)^{1/4}. \quad (19)$$

The coefficients C_m occurring in equation (18) are given by the general formula:

$$C_m = \frac{1}{\sigma^{(m-3)/4}} \frac{f_0{}^{(m)}}{(m+1)!} \left(\frac{4!}{f'''_0}\right)^{(m+1)/4}. \quad (20)$$

Again expanding the exponential term in the integrand of equation (18) gives a series for E_{sep} in terms of gamma functions and the coefficients C_m. Using the values $f'''_0 = 0{\cdot}1988$ and $\Gamma(\frac{1}{4}) = 3{\cdot}62561$ gives finally for E_{sep} the asymptotic expansion:

$$E_{sep} = 3{\cdot}62561 + \frac{0{\cdot}084049}{\sigma} - \frac{0{\cdot}005907}{\sigma^2} - \frac{0{\cdot}0001405}{\sigma^3} + 0(\sigma^{-4}). \quad (21)$$

As has been noted the coefficients in this series differ from the values given by Merk [4]. The series also contains one more term than was given by Merk.

3.3 *Asymptotic behaviour at high σ*

It is clear from equation (13) that for high values of σ the integral E is constant. From equation (10) therefore, (b'_0/B) is proportional to $\sigma^{1/3}$. This is a well-known relationship which was exploited when plotting some of the figures given in Paper 3. In order to bring closer together curves for different values of σ, the group $\sigma^{-1/3}(b'_0/B)$ was plotted against the driving force B.

This relationship does not, however, hold for the separation point. From equations (19) and (21) it may be seen that for this case when σ is large, (b'_0/B) is proportional to $\sigma^{1/4}$.

These asymptotic relationships will be used later to include on the same figures curves relating to a wide range of σ.

4. FUNCTIONS OBTAINED FROM THE ASYMPTOTIC SERIES

4.1 *Introduction*

The asymptotic formulae given in Table 2 for evaluating (b'_0/B) should only be used for $\sigma \geqslant 0{\cdot}5$ since the accuracy decreases rapidly as σ decreases below this value.

Using these formulae values of (b'_0/B) have been calculated for wide ranges of β and σ; the results are given in Table 3.

The range of main-stream pressure gradient included in Table 3 covers the values encountered in a wide range of practical problems. It includes the point of separation for two-dimensional flow ($\beta = -0{\cdot}1988$), the case of zero pressure gradient ($\beta = 0$), the stagnation point for axially symmetric, three-dimensional flow (corresponding to $\beta = 0{\cdot}5$; see Paper 1), the stagnation point for two-dimensional flow ($\beta = 1{\cdot}0$) as well as some more highly accelerated main-streams than the last.

For application to problems involving heat transfer calculations, when σ is the Prandtl number, the table covers the case of most gases, which have a Prandtl number near 0·7, as well as the majority of liquids. The Prandtl number for water, for example, ranges from about 7 at room temperature to about 2·0 at 200°F. Typical values for a more viscous fluid like engine oil would be $\sigma = 10\,000$ at room temperature to $\sigma = 1000$ at 150°F. Fluids such as liquid metals, which have a high thermal conductivity and therefore a low Prandtl number (e.g. for mercury $\sigma = 0{\cdot}02$), are not however included.

Fluids will rarely have values of σ exactly equal to those given in Table 3 but the intervals in the

Table 3. Values of the wall gradient (b_0'/B) as a function of the parameter β and the fluid property σ

σ \ β	—0·1988	—0·1	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·8	1·0	1·4	2·0	3·0	4·0	5·0
0·5	0·2692	0·3443	0·3667(+3)	0·3811	0·3919	0·4003	0·4071	0·4130	0·4180	0·4263	0·4332	0·4437	0·4563	0·4725	0·4901	0·5111
0·6	0·2833	0·3669	0·3917(0)	0·4077	0·4197	0·4291	0·4367	0·4434	0·4491	0·4585	0·4663	0·4781	0·4918	0·5081	0·5231	0·5388
0·7	0·2957(+2)	0·3869(—4)	0·4139(+2)	0·4315	0·4445	0·4548	0·4633	0·4706(+1)	0·4769	0·4872	0·4958(0)	0·5089	0·5238	0·5409	0·5552	0·5686
0·8	0·3067	0·4049	0·4341(—1)	0·4530	0·4671	0·4782	0·4873	0·4953	0·5021	0·5134	0·5227	0·5370	0·5531	0·5712	0·5855	0·5980
0·9	0·3167	0·4214	0·4525(0)	0·4727	0·4878	0·4997	0·5095	0·5180	0·5253	0·5374	0·5475	0·5629	0·5799	0·5994	0·6141	0·6263
1·0	0·3258	0·4367	0·4696(0)	0·4910	0·5070	0·5196	0·5300	0·5390	0·5468	0·5598	0·5705	0·5869	0·6053	0·6258	0·6410	0·6534
1·1	0·3343	0·4509	0·4856(—2)	0·5081	0·5249	0·5382	0·5492	0·5587	0·5670	0·5806	0·5920	0·6094	0·6288	0·6506	0·6665	0·6791
1·2	0·3421	0·4643	0·5005	0·5241	0·5417	0·5557	0·5672	0·5773	0·5859	0·6003	0·6122	0·6306	0·5611	0·6740	9·6906	0·7036
1·4	0·3564	0·4889	0·5281	0·5536	0·5728	0·5880	0·6005	0·6114	0·6209	0·6365	0·6496	0·6697	0·6921	0·7173	0·7354	0·7494
1·6	0·3692	0·5111	0·5531	0·5804	0·6010	0·6172	0·6307	0·6424	0·6526	0·6695	0·6835	0·7053	0·7295	0·7568	0·7764	0·7914
1·8	0·3808	0·5314	0·5760	0·6050	0·6268	0·6441	0·6585	0·6709	0·6818	0·6998	0·7147	0·7380	0·7639	0·7933	0·8143	0·8304
2·0	0·3914	0·5503	0·5972(+1)	0·6278	0·6508	0·6691	0·6842	0·6974	0·7088	0·7279	0·7437	0·7684	0·7959	0·8272	0·8496	0·8667
2·5	0·4148	0·5923	0·6446	0·6788	0·7044	0·7248	0·7418	0·7565	0·7693	0·7907	0·8086	0·8364	0·8676	0·9032	0·9288	0·9484
3·0	0·4348	0·6289	0·6860	0·7232	0·7512	0·7735	0·7920	0·8081	0·8222	0·8456	0·8652	0·8959	0·9303	0·9698	0·9983	1·0201
3·5	0·4523	0·6616	0·7228	0·7628	0·7929	0·8169	0·8369	0·8542	0·8694	0·8947	0·9159	0·9491	0·9864	1·0294	1·0605	1·0844
4·0	0·4680	0·6911	0·7563	0·7988	0·8308	0·8564	0·8776	0·8961	0·9123	0·9393	0·9619	0·9974	1·0373	1·0837	1·1171	1·1430
4·5	0·4823	0·7183	0·7871	0·8319	0·8657	0·8926	0·9151	0·9346	0·9517	0·9803	1·0042	1·0419	1·0842	1·1336	1·1693	1·1969
5·0	0·4954	0·7435	0·8156	0·8626	0·8980	0·9263	0·9498	0·9703	0·9882	1·0183	1·0434	1·0831	1·1278	1·1799	1·2177	1·2471
6	0·5189	0·7892	0·8673	0·9183	0·9566	0·9873	1·0129	1·0351	1·0546	1·0873	1·1147	1·1580	1·2068	1·2641	1·3057	1·3382
7	0·5396	0·8299	0·9135(0)	0·9680	1·0090	1·0418	1·0692	1·0930	1·1139	1·1490	1·1784	1·2249	1·2775	1·3394	1·3845	1·4197
8	0·5582	0·8670	0·9554	1·0131	1·0566	1·0914	1·1204	1·1457	1·1678	1·2050	1·2362	1·2857	1·3418	1·4078	1·4561	1·4939
9	0·5750	0·9010	0·9940	1·0546	1·1003	1·1369	1·1674	1·1940	1·2173	1·2566	1·2894	1·3415	1·4008	1·4707	1·5219	1·5621
10	0·5905	0·9325	1·0298(0)	1·0932	1·1409	1·1792	1·2111	1·2389	1·2633	1·3044	1·3388	1·3935	1·4557	1·5291	1·5831	1·6255
15	0·6540	1·0646	1·1796(0)	1·2546	1·3110	1·3562	1·3941	1·4270	1·4559	1·5047	1·5457	1·6111	1·6855	1·7741	1·8396	1·8913
20	0·7031	1·1696	1·2988	1·3829	1·4463	1·4971	1·5396	1·5766	1·6092	1·6641	1·7103	1·7841	1·8684	1·9691	2·0438	2·1030
25	0·7436	1·2582	1·3994	1·4913	1·5605	1·6160	1·6625	1·7029	1·7385	1·7987	1·8492	1·9302	2·0227	2·1337	2·2162	2·2817
30	0·7784	1·3355	1·4873	1·5860	1·6603	1·7199	1·7698	1·8133	1·8515	1·9162	1·9706	2·0578	2·1576	2·2775	2·3668	2·4379
40	0·8366	1·4676	1·6373	1·7475	1·8306	1·8971	1·9530	2·0016	2·0443	2·1168	2·1778	2·2756	2·3878	2·5229	2·6239	2·7045
50	0·8847	1·5790	1·7639	1·8839	1·9743	2·0468	2·1076	2·1605	2·2071	2·2861	2·3526	2·4594	2·5820	2·7301	2·8409	2·9295
100	1·0523	1·9828	2·2239	2·3784	2·4954	2·5893	2·6681	2·7367	2·7972	2·8998	2·9863	3·1257	3·2861	3·4808	3·6273	3·7450
200	1·2515	2·4912	2·8010	3·0011	3·1517	3·2725	3·3739	3·4623	3·5402	3·6726	3·7843	3·9645	4·1725	4·4260	4·6173	4·7716
500	1·6442	3·3713	3·8018	4·0792	4·2878	4·4552	4·5958	4·7184	4·8265	5·0103	5·1655	5·4164	5·7065	6·0613	6·3302	6·5477
1000	1·8716	4·2403	4·7901	5·1438	5·4097	5·6231	5·8024	5·9586	6·0965	6·3310	6·5291	6·8497	7·2208	7·6757	8·021	8·301
2000	2·2258	5·3350	6·0352	6·4851	6·8231	7·0945	7·3224	7·5211	7·6964	7·995	8·247	8·655	9·128	9·709	10·151	10·509
5000	2·7988	7·2304	8·1910	8·808	9·270	9·642	9·954	10·226	10·467	10·876	11·221	11·781	12·431	13·229	13·837	14·331
10000	3·3284	9·102	10·320	11·101	11·687	12·158	12·553	12·898	13·202	13·720	14·158	14·868	15·692	16·705	17·477	18·105
20000	3·9581	11·461	13·003	13·991	14·732	15·328	15·828	16·264	16·649	17·304	17·859	18·757	19·801	21·805	22·063	22·860

Figures in brackets give differences in the last digit between the values given in the table and some exact values found in the literature. The accuracy is lower in the region marked off by broken lines.

parameters β and σ are believed to be convenient for interpolation over most parts of the table. This is clearly not true for decelerated flow approaching the conditions when the velocity layer separates. Although such interpolation would give the sort of accuracy needed for many practical problems, higher accuracy may sometimes be needed for some value of σ not given. It is then a simple matter to calculate new values of (b_0'/B) from the series given in Table 2.

4.2 *Accuracy of Table* 3

The series from which the values of (b_0'/B) given in Table 3 were calculated are asymptotically accurate for high values of σ. For sufficiently large σ therefore the figures should be correct to one place in the last digit. For $\sigma \leqslant 1$ however the accuracy may not be so high.

Some idea of the accuracy of the values may be obtained by comparing with exact solutions quoted in the literature. This is done in Table 3 by giving in brackets the differences in the final digit between the present values and exact solutions given in the literature. The exact values were taken from Mickley *et al.* [7], Merk [4], Livingood and Donoughe [8] and Howe and Mersman [9]. It should be realised, however, that exact solutions quoted in the literature are found to disagree with each other by up to three units in the fourth significant digit.

Exact calculations made since Table 3 was drawn up, using an entirely different method, have largely confirmed the figures in the table. Many values can be in error by one unit in the last place due to rounding off. This is, in fact, the case. Apart from these, however, the only serious errors in the table are for $\sigma = 0{\cdot}5$ and $0{\cdot}6$ but even so only amount to 3 units in the last significant digit. The results of exact calculations are to be published elsewhere.

No exact calculations have, however, been made for the region marked off in the table as inaccurate. Values in this region are almost certainly inaccurate.

4.3 *Other functions evaluated from* (b_0'/B)

It was shown in Paper 3 that a number of other functions related to the b-boundary layer may be evaluated from (b_0'/B). For ease of reference, and because many of the formulae take on a special form when no mass flows through the interface, the relevant formulae are listed below:

$$\frac{\Delta_4^2}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \frac{\beta}{(b_0'/B)^2} \tag{22}$$

$$\frac{u_G}{\nu}\frac{\mathrm{d}\Delta_4^2}{\mathrm{d}x} = \frac{2(1-\beta)}{(b_0'/B)^2} \tag{23}$$

$$\frac{\Delta_2^2}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \frac{\beta}{\sigma^2}\left(\frac{b_0'}{B}\right)^2 \tag{24}$$

$$\frac{u_G}{\nu}\frac{\mathrm{d}\Delta_2^2}{\mathrm{d}x} = \frac{2(1-\beta)}{\sigma^2}\left(\frac{b_0'}{B}\right)^2 \tag{25}$$

$$\frac{\Delta_2}{\Delta_4} = \frac{1}{\sigma}\left(\frac{b_0'}{B}\right)^2 \tag{26}$$

$$g^* = \frac{\gamma}{\Delta_4} = \gamma\left(\frac{1}{\nu\beta}\frac{\mathrm{d}u_G}{\mathrm{d}x}\right)^{1/2}\left(\frac{b_0'}{B}\right) \tag{27}$$

$$X = \frac{\Delta_4\delta_4}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \frac{\beta}{f_0''}\frac{1}{(b_0'/B)} \tag{28}$$

$$Y = \frac{1}{\gamma}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\Delta_4^3\left(\frac{u_G}{\delta_4}\right)^{3/2}\right\} = \frac{3}{2}\frac{\sigma f_0''}{(b_0'/B)^3} \tag{29}$$

$$W = \frac{\Delta_2^{1/2}\delta_4^{3/2}}{\nu}\frac{\mathrm{d}u_G}{\mathrm{d}x} = \frac{\beta(b_0'/B)^{1/2}}{\sigma^{\frac{1}{2}}(f_0'')^{3/2}} \tag{30}$$

$$Z = \frac{1}{\gamma}\left(\frac{\delta_4}{u_G}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}(u_G\Delta_2)^{3/2} = \frac{3}{2}\frac{(b_0'/B)^{3/2}}{(\sigma f_0'')^{1/2}}\,. \tag{31}$$

In the literature many functions are written in terms of the distance x of the point in question from the start of the boundary layer. In terms of the present formulation of similar solutions some of these functions are:

$$Nu/Re^{1/2} = \frac{1}{(2-\beta)^{1/2}}\left(\frac{b_0'}{B}\right) \tag{32}$$

$$\frac{\Delta_4 Re^{1/2}}{x} = \frac{(2-\beta)^{1/2}}{(b_0'/B)} \tag{33}$$

$$\frac{\Delta_2 Re^{1/2}}{x} = \frac{(2-\beta)^{1/2}}{\sigma}\left(\frac{b_0'}{B}\right) \tag{34}$$

where the Nusselt number Nu and the Reynolds number Re have local values.

5. CURVES OF FUNCTIONS OBTAINED FROM TABLE 3

5.1 *General*

Curves have been drawn showing the relationships between many of the functions occurring on the left-hand sides of equations (22) to (31). Some interesting features about the curves will be indicated but their application for calculating functions in non-similar boundary layers will not be discussed. This is to be done in other papers in the series.

In Papers 1 and 2 it was shown that, in order to use the "similar" solutions to the velocity equation to obtain information about non-similar boundary layers, tables or curves were needed showing the variation of a quantity denoted by F_2 with the parameter $(\delta_2^2/\nu)(\mathrm{d}u_G/\mathrm{d}x)$. The latter is clearly a pressure gradient parameter related to the momentum boundary layer thickness δ_2 and F_2, which is $(u_G/\nu)(\mathrm{d}\delta_2^2/\mathrm{d}x)$, is a measure of the rate of growth of the momentum thickness with distance x.

Figures 1(a, b) and 2(a, b) give analogous relationships for thicknesses associated with the b-boundary layer. The relevant functions have been multiplied by an appropriate power of σ in order to bring closer together curves for different values of σ. Early forms of these curves for one value of σ, namely $\sigma = 0{\cdot}7$, were given by Smith and Spalding [10].

5.2 *Figure* 1(a)

This figure, which applies to the conduction thickness Δ_4, shows the relationship for accelerated and slightly decelerated flows. Because the wall gradient (b_0'/B) is almost exactly proportional to $\sigma^{1/3}$ when σ is large, the multiplying factor in this figure is $\sigma^{2/3}$. As σ becomes very

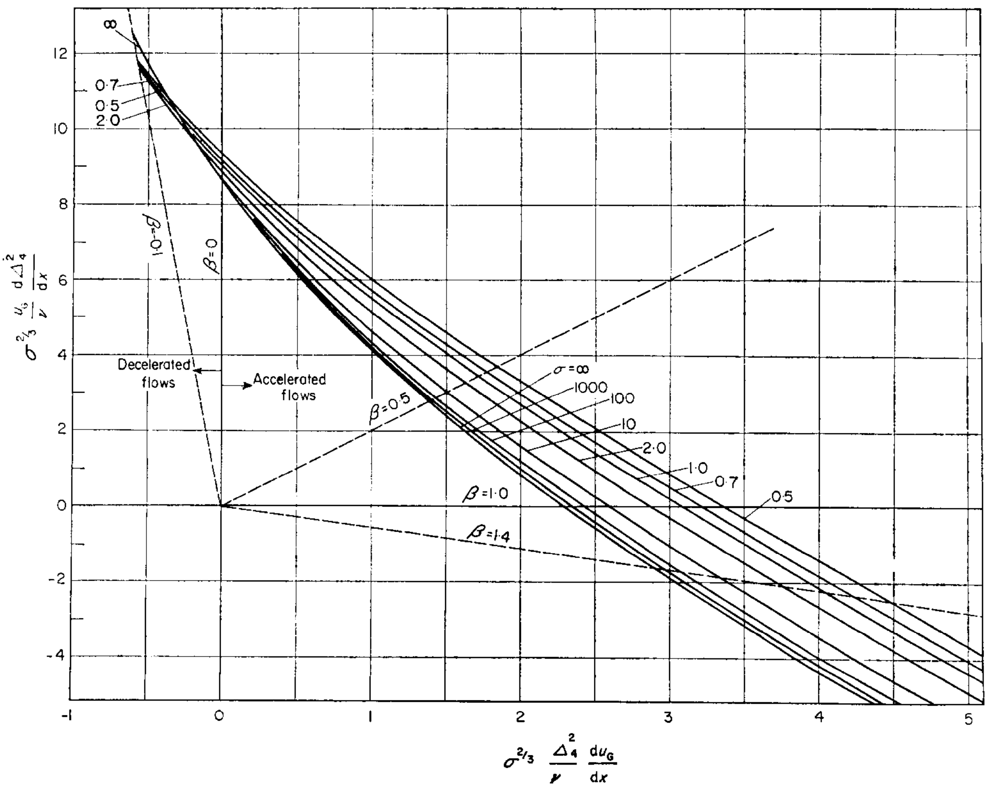

FIG. 1(a) Conduction thickness Δ_4, accelerated and slightly decelerated flows.

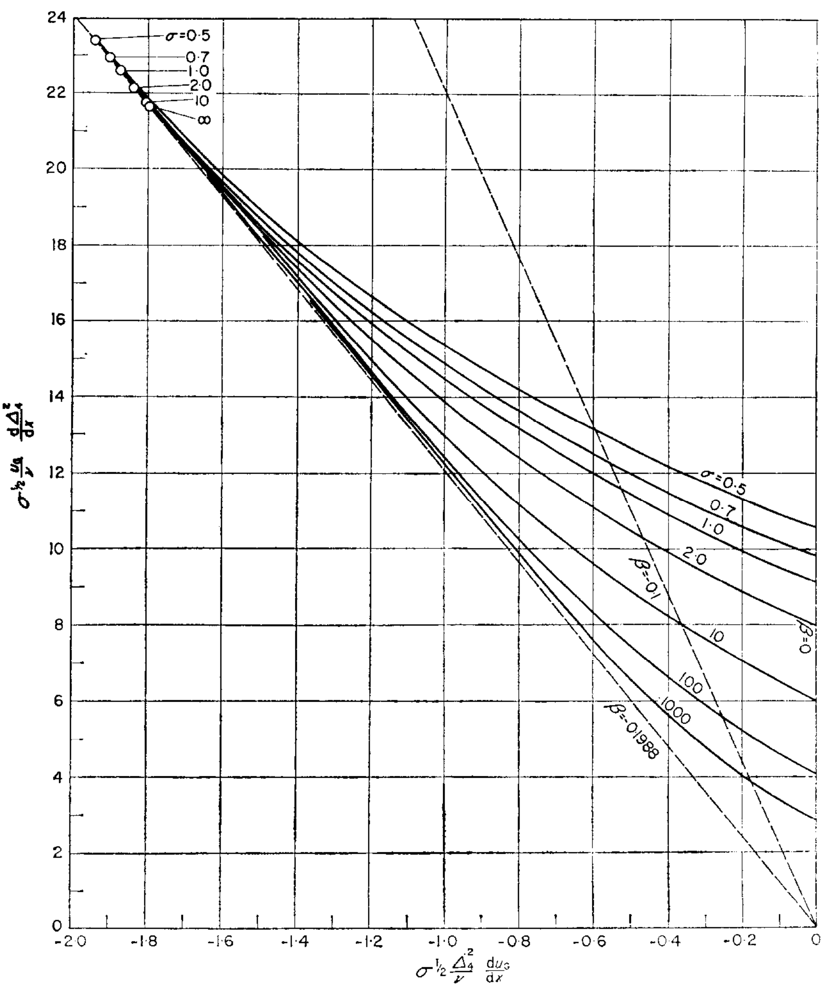

FIG. 1(b) Conduction thickness Δ_4, decelerated flows. ○ Separation points.

large the curves clearly tend to the asymptotic curve shown as $\sigma = \infty$.

It is clear from the curves for decelerated flows shown in this figure, which extend as far as $\beta = -0{\cdot}1$, that the wall gradient (b_0'/B) is not proportional to $\sigma^{1/3}$ in the lower part of the range covered. Hence the cross-over of the curves in this region. For more highly decelerated flows the curves diverge rapidly, particularly those for high σ.

Because of the scale of this figure it was convenient to omit parts corresponding to highly accelerated flows. Since the lines are only slightly curved in this region it is easy to see how they continue downwards and to the right.

5.3 *Figure* 1(b)

In section 3.3 it was noted that for the separation point (b_0'/B) is proportional to $\sigma^{1/4}$. The curves corresponding to Fig. 1(a), but for decelerated flows, have therefore been drawn using the multiplying factor $\sigma^{1/2}$. The separation points for a number of values of σ are shown on this figure.

There is some uncertainty about the shapes of these curves as only three values of β could be used from the present results. The curve for

$\sigma = 0{\cdot}7$ was drawn in using the results given by Livingood and Donoughe [8]. The other curves were then made to have the same general shape as this between $\beta = -0{\cdot}1$ and $\beta = -0{\cdot}1988$.

For $\sigma = \infty$ only the separation point is included because for both $\beta = 0$ and $\beta = -0{\cdot}1$ the relevant point is at the origin.

5.4 *Figure* 2(a)

This figure, which relates to the convection thickness Δ_2, corresponds to Fig. 1(a). The curves have not been extended into the region of decelerated flow, however, as the shapes of the curves could not be judged as was done for the earlier figure.

It was again convenient to omit the region of highly accelerated flows.

5.5 *Figure* 2(b)

This figure relates to the convection thickness Δ_2 for decelerated flows. The scale of the figure did not allow the curves for $\sigma > 1000$ to be included and parts of the curves for $\sigma \geqslant 10$ had to be omitted because their shapes could not be judged with any certainty from that for $\sigma = 0{\cdot}7$.

Again a few separation points are indicated.

5.6 *Figures* 3 *and* 4

In suggesting an approximate method for calculating rates of heat transfer between any laminar fluid stream and a solid surface, Spalding [11] set up differential equations which could be integrated numerically in a fairly straightforward manner. Referring to expressions defined in equations (28) to (31), the differential equation

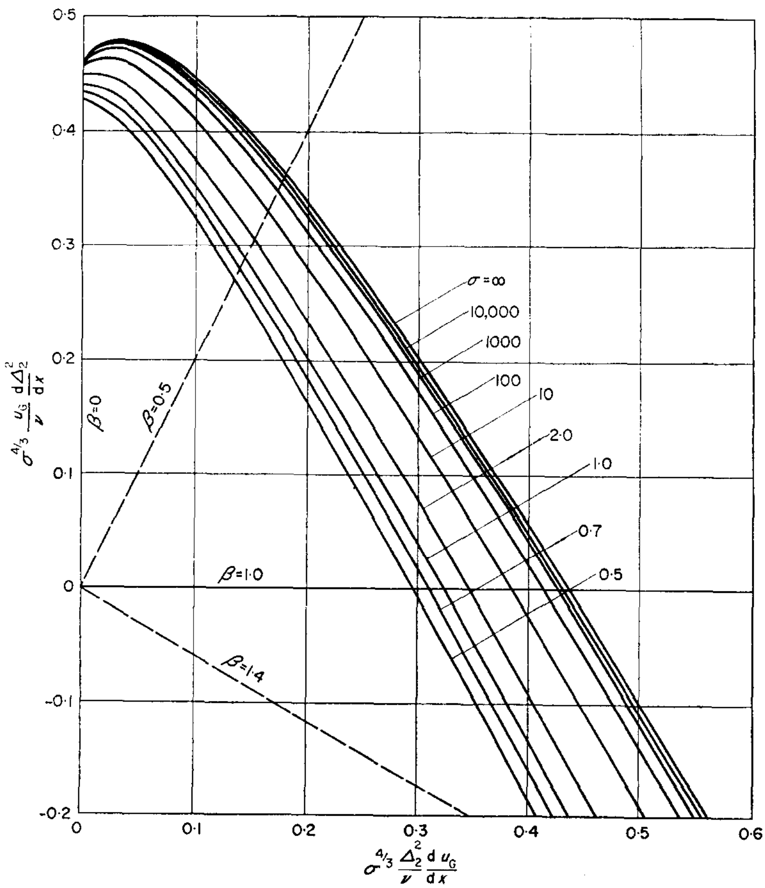

FIG. 2(a) Convection thickness Δ_2 accelerated flows.

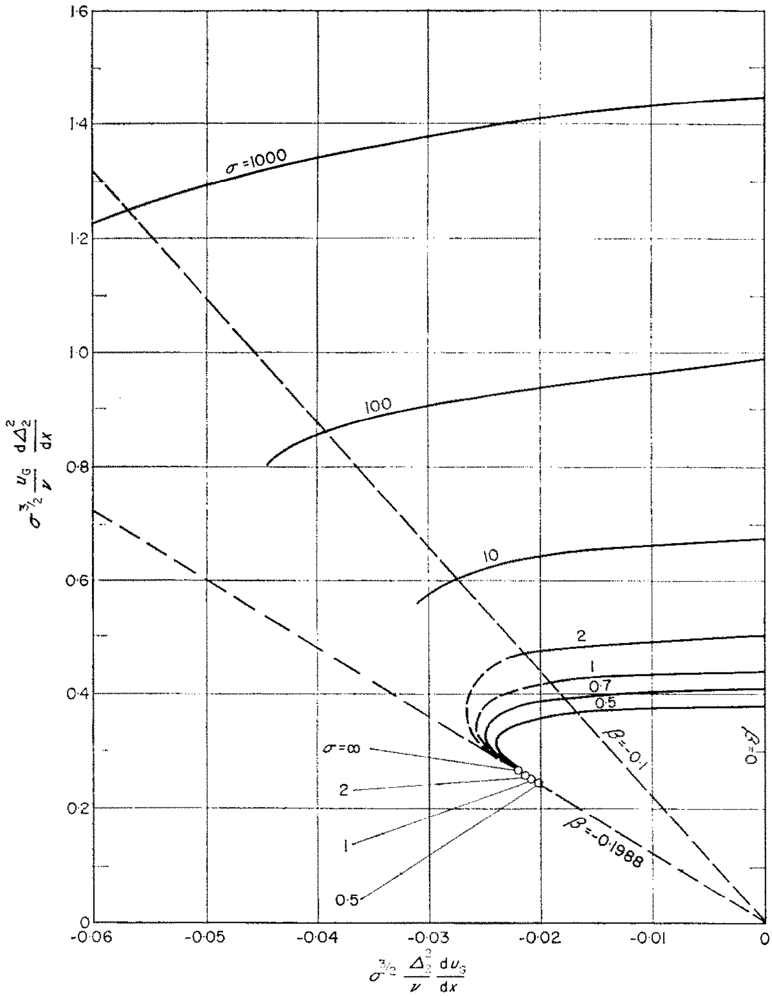

FIG. 2(b) Convection thickness Δ_2 decelerated flows. ○ Separation points.

for evaluating the conduction thickness Δ_4 was of the form:

$$Y = Y(X) \tag{35}$$

and for the convection thickness Δ_2 the equation was:

$$Z = Z(W). \tag{36}$$

In these equations the functions Y and Z are measures of the rate of growth of the relevant boundary layer thickness with distance x along the wall, and the functions X and W are curvature parameters which are a measure of the amount by which the b-boundary layer penetrates into regions where the velocity layer is appreciably curved. The forms of the functions Y and Z were obtained from the exact "similar" solutions to the boundary layer equations.

When plotting Y and Z as functions of X and W respectively, it was found that exact similar solutions for a wide range of Prandtl number were close to a single curve. Figs. 3 and 4 show these relationships obtained from the present results. Curves for only a few values of σ are drawn on these figures because such curves are very close together on this scale.

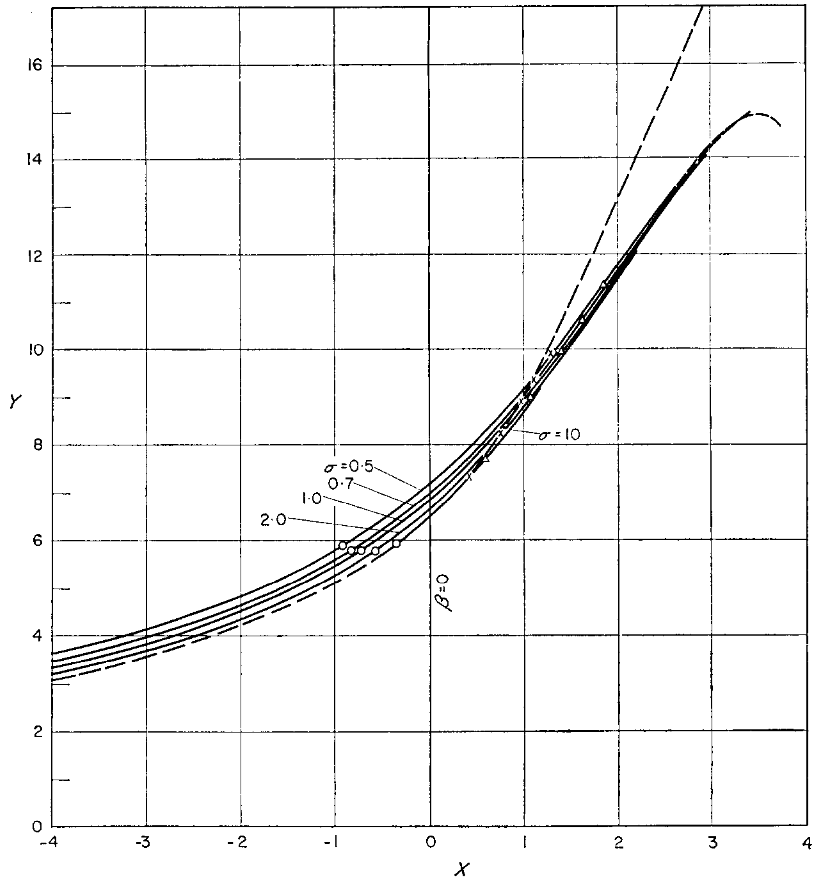

FIG. 3. Y as a function of X.

— — — σ large (from Paper 3).
———— present solutions.
- - - - - inaccurate region.

○ $\beta = -0{\cdot}1$
× $\beta = 0{\cdot}5$
△ $\beta = 1{\cdot}0$.

When $\sigma \geqslant 0{\cdot}5$, therefore, the assumption of a single curve for the relationships given in equations (35) and (36) would introduce only a small error in the estimation of boundary layer thicknesses for accelerated and slightly decelerated flows. For conditions approaching separation, however, the error would increase rapidly. These conditions occur for large negative values of the abscissae X and W, beyond the ranges of Figs. 3 and 4. The separation points themselves correspond to $X = -\infty$, $Y = 0$ and $W = -\infty$, $Z = \infty$ respectively.

In Figs. 3 and 4 the line for large σ was obtained from the approximate results given in Paper 3.

It should be noted that when $\sigma < 1{\cdot}0$ and $\beta > 2{\cdot}0$ the present results are inaccurate. This inaccurate region has already been indicated in Table 3. This shows up in Figs. 3 and 4 as a reversal in the slopes of the curves for large values of the abscissa.

In Fig. 3 all the results for $\sigma > 20$, except that for the separation point, are close to the point $X = 0$, $Y = 6{\cdot}41$, the well-known solution obtained by Lighthill [12].

The points on Figs. 3 and 4 give a few values of the parameter β. The curves from the present solutions have been extended beyond $\beta = -0{\cdot}1$ by giving them the same shape as those for $\sigma = 0{\cdot}7$ and σ-large whose shapes were known.

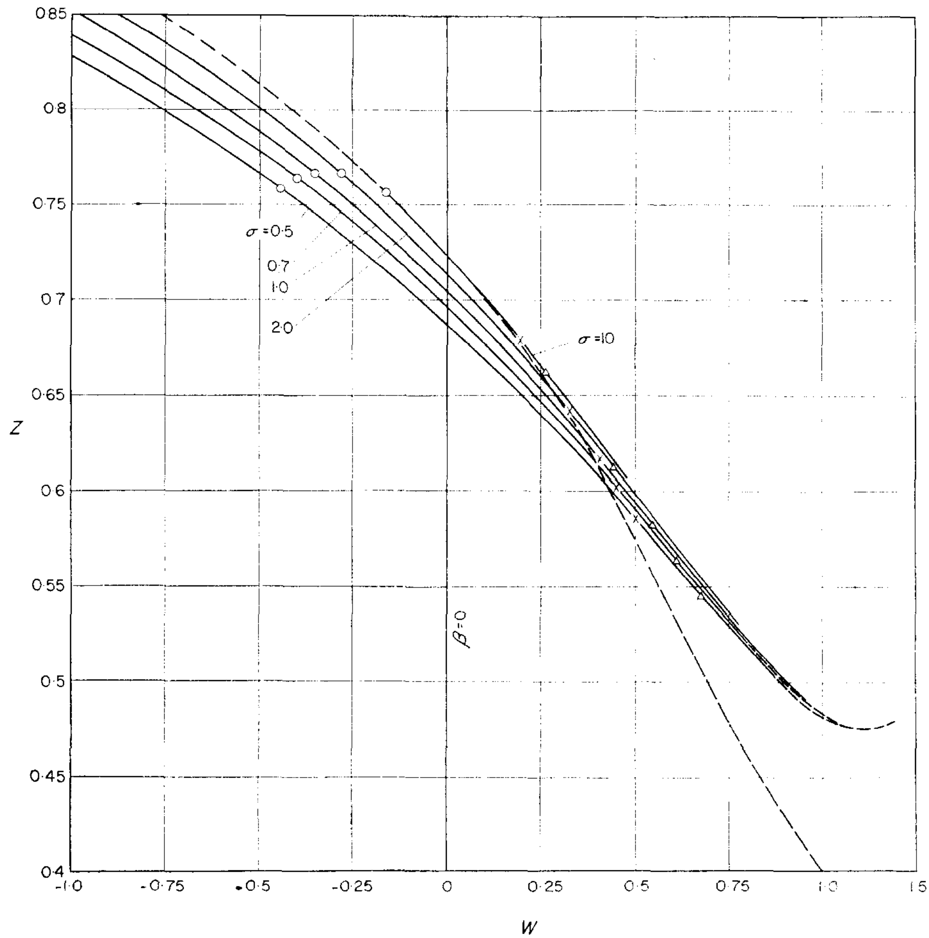

FIG. 4. Z as a function of W.

— — — σ large (from Paper 3). ○ $\beta = -0{\cdot}1$
——— present solutions. × $\beta = 0{\cdot}5$
- - - - - inaccurate region. △ $\beta = 1{\cdot}0$.

6. APPROXIMATE METHOD NEAR SEPARATION

It has been seen that, if it is greater than 0·5, the Prandtl/Schmidt number has only a small effect when boundary layer problems are formulated in terms of the functions X, Y, W and Z. It is, of course, for the same reason that the asymptotic series of Table 2 give good accuracy. The close connection between the present results and this approximate formulation may be seen by comparing, say, equations (10) and (29). The rate of growth parameter Y is related to the integral E of the present paper by:

$$Y = \frac{E^3}{3}. \tag{37}$$

This immediately suggests that near the point of separation the parameter Y should be replaced by Y_{sep} given by:

$$Y_{sep} = \frac{1}{\gamma}\left(\frac{\partial^2 u}{\partial y^2}\right)_0^{-1/3} \frac{d}{dx}\left\{\Delta_4^4 \left(\frac{\partial^2 u}{\partial y^2}\right)_0^{4/3}\right\}$$
$$= \frac{4}{3}\,\frac{\sigma f'''_0}{(b'_0/B)^4} \tag{38}$$

where the second form is written in terms of "similar" solutions. The corresponding curvature parameter is:

$$X_{\text{sep}} = \frac{(\partial^4 u/\partial y^4)_0 \Delta_4^2}{(\partial^2 u/\partial y^2)_0} = \frac{f_0^{\text{v}}}{(f_0''')(b_0'/B)^2} \qquad (39)$$

the second form again being in terms of "similar" solutions. For the case of no mass transfer, the gradients of f occurring in the last expression are given by $f_0''' = -\beta$ and $f_0^{\text{v}} = (2\beta - 1)(f_0'')^2$.

It should be noted that $(\partial^3 u/\partial y^3)_0$ cannot be used in the numerator of X_{sep} because it is identically zero for all β when no mass flows through the wall. For very large σ it may be shown that Y_{sep} tends to the value 21·599.

In the same way other functions can be derived to replace W and Z near the point of separation.

7. CONCLUSIONS

(a) Using the formulae and coefficients given in Table 2 and the values of f_0'' of Table 1, the wall gradient (b_0'/B) can be evaluated accurately for a wide range in the parameter β and for any value greater than 0·5 of the Prandtl/Schmidt number σ.

(b) From values of (b_0'/B) many other functions associated with the b-boundary layer can also be evaluated from the formulae of section 4.3.

(c) Using a different method of calculation more accurate results than those in Table 3 have now been obtained. They include results for values of β, in particular many negative values, not covered in the present paper as well as a range of low values of σ. These are to be published elsewhere.

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